On Controller & Capacity Allocation Co-Design for Networked Control Systems [1]

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Abstract

This paper presents a framework for examining joint optimal channel-capacity allocation and controller design for networked control systems using store-and-forward networks in a discrete-time linear time-invariant setting. The resultant framework provides a synthesis procedure for designing distributed linear control laws for capacity-constrained networks taking the allocation of the capacity within the network into account.

Key words: networked control systems, optimal design, channel capacity

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For networked systems with a large number of nodes (sensors, controllers and actuators) or systems where mutual communication amongst all nodes is difficult or slow, networked control system design presents significant challenges including considerations of the choice of network topology (inter-node connectivity and inter-node delay), allocation of inter-node channel capacity and total network capacity in addition to the controller design under these communication constraints. These considerations are similar to the guiding principles of *distributed* control system design.

The philosophy of distributed system design is to dispense with the notion that the system in question can be controlled through a single centralized control law and distribute the control task across a number of communicating

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controllers that, together, achieve the desired behavior. Our focus will be on stabilization.

Communication over networks entails a cost in the form of delays and capacity requirements needed to achieve stability. In various settings, the results presented in Braslavsky et al. [2004], Nair et al. [2004], Nair and Evans [2004], Nair et al. [2007] have established the minimum data rates (channel capacities) needed to stabilize linear systems [2]. For discrete-time systems, both Braslavsky et al. [2004] and Nair et al. [2004] establish that the minimum data rate needed to stabilize a linear plant of the form

$$x^{+} = Ax + Bu \quad y = Cx \tag{1}$$

is given by $R > C$, where

$$C = \sum_{|\eta_i| \geq 1} \log_2 |\eta_i| \text{ bits per second} \tag{2}$$

and η_i are the eigenvalues of A.

It was shown that (2) is necessary and sufficient for the existence of a coding and control law that gives exponential convergence of the state to the origin from a random initial state. The primary observation in Braslavsky et al. [2004] was that the channel may impose a bit rate limitation for signals in the control loop through a constraint on the signal to noise ratio (SNR) for the communication channel. SNR-constrained channels were considered Braslavsky et al. [2004] and all pre- and post- signal processing involved in the communication link was restricted to LTI filtering and digital-to-analogue and analogue-to-digital type operations. Hence, the communication link reduces to the noisy channel itself. By application of the Shannon-Hartley Theorem, Braslavsky et al. [2004] recovers the bound (2) for discrete-time systems and presents analogous bounds for continuous-time systems.

From an NCS point of view, these results are phrased in terms of the information flow from a controller to a monolithic actuator and not the information flows between nodes (which directly measure plant states or outputs) and controllers. Using an SNR characterization of channel capacity, this paper will present a discrete-time framework for optimal distributed (state-feedback) controller design specifically taking link-to-link capacity and network structure into account.

The signal in the per-channel SNR ratio is the state x_i and we assume the (local) controller directly actuates the plant, hence there are no capacity con-

[2] Nonlinear results are established in Liberzon and Hespanha [2005], Nair and Evans [2004] but we restrict our attention to linear results where the bounds have been shown to be necessary and sufficient and the calculations of the bounds is tractable.

straints in communicating the control signal to the plant. Our design framework establishes the lowest SNR needed to communicate individual states to *any controller* u_j but without reference to the path traversed in the network; the relevant SNR, and hence, capacity, for each state is maintained along path components from the node measuring the state to (each) controller. The next phase of the design process is solving for the optimal allocation of the capacities within directed graphs. Regarding the node measuring state x_r as a producer of a commodity r and each controller dependent on x_r as a consumer, the capacity allocation problem can be posed as a multi-commodity graph flow problem that is solved efficiently through linear programming (LP).

The design process posited entails:

(a) solving for the optimal linear control law compatible with the information structure imposed by the network graph;
(b) solving for end-to-end (state-to-controller) capacities; and
(c) determining the optimal allocation of capacities within the network, thereby establishing the capacities along each edge of directed graphs describing the network.

1 Information Structures for Control

An information structure for a control system is, essentially, a description of the dependencies of the control law on observable system information. Henceforth, our discussion will be restricted to stochastic discrete-time systems with state feedback:

$$z^+ = \Phi z + \Gamma u + w\,, \tag{3}$$

and cost functions of the form

$$v = \mathbf{E}|Cz + Du|^2\,, \tag{4}$$

where $z(0)$ is Gaussian and zero-mean and w is zero-mean white Gaussian noise with unit variance (identity covariance), hence, "dependence" is equivalent to non-zero covariance between respective random variables. As an example of an information structure, neglecting channel noise and writing $u_j(k) = \mu_j(z_1(k), z_2(k), z_4(k-3))$, we are implying that control u_j depends only the current values of z_1, z_2 and z_4 delayed by three time-steps (and no other random variables). We can also express this information structure in the form of a directed graph as in Figure 1.

Dependency of $u_j(k)$ on state $z_i(k-\Delta_{ji})$ in an information structure is equivalent to connectivity along a directed path in the graph with a path-length of Δ_{ji}. Note that dependencies of $u_j(k)$ on $z_i(k)$ corresponds to collocation of

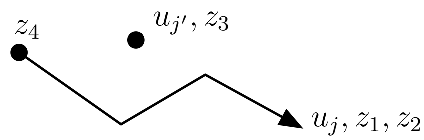


Fig. 1. Directed graph representation of an information structure.

u_j and z_i at a graph vertex and independence of u_j and z_i is equivalent to asserting that there does not exist a directed path from z_i to u_j.

Suppose now that instead of representing an information structure, Figure 1 represents a network topology. That is, the directed paths represent existence of a directed *network* communication channel between elements of the path (with delays commensurate with path-length). In a directed sense, each connected component of the graph is a representation of *allowable* dependencies of controllers on incident state variables in that graph component.

In the discussion of information structures and network-connectivity, it becomes evident that the plant itself may be used as a communication channel leading to the possibility of complex nonlinear control laws that "signal" information through the plant. The optimal design problem may quickly become intractable if the admissible information structures are not carefully characterized, as demonstrated by the so-called Witsenhausen counterexample in stochastic optimal control Witsenhausen [1968] where it was shown that the optimal strategies *may not be linear* in spite of the apparent simplicity of the problem and Gaussian signals. Notwithstanding nonlinear laws proposed in Martins [2006] and references cited therein, progress towards tractability of the synthesis problem can be made by eliminating information structures that create signaling incentives. Progress has been made in rendering the synthesis problem tractable and, in particular, convex in Ho and Chu [1972], Rotkowitz and Lall [2002], Bamieh et al. [2005], Bamieh and Voulgaris [2005], Rantzer [2006] in various settings.

Our presentation builds on that of Rantzer [2006] where the (infinite-horizon) distributed control synthesis problem is solved by expressing information structure constraints as covariance constraints and restricting the set of admissible information structures to those that eliminate the "signaling incentive". The synthesis problem is shown to be convex and solutions can efficiently be obtained by solving a linear matrix inequality (LMI); e.g., through interior-point methods described in Boyd et al. [1995]. The two central observations in Rantzer [2006] are that for systems of the form (3) employing static state feedback, we can express information structure constraints as controller-process noise covariance constraints; and, for each (sub)controller u_j, either the network propagates information at least as fast as the plant does, or, the network propagates information that is not communicable through the plant. With these observations, the optimal synthesis problem for the given information structure becomes

$$\begin{aligned} \text{minimize} \quad & \mathbf{E}\,|Cz + Du|^2 \quad \text{subject to} \\ & \mathbf{E}\left\{x(k)^T R_j u(k)\right\} \quad 1 \leq j \leq J \end{aligned}$$

for appropriately chosen matrices R_j and $x = \begin{bmatrix} z(k)^T \ w(k-1)^T \ \ldots \ w(k-N)^T \end{bmatrix}^T$ and $u(k) = \begin{bmatrix} u_1(k)^T \ \ldots \ u_J(k)^T \end{bmatrix}^T$, where N is the length of the largest delay from any measurement to any subcontroller. By careful selection of a cost function that penalizes per-state SNR and decoupling the notion of an information structure from that of network-connectivity, the proceeding section establishes the basic result for the optimal capacity-control co-design framework of this paper.

2 State Feedback With Covariance Constraints

The following result is a general theorem on optimal state-feedback design with covariance constraints for systems of the form $x^+ = Ax + Bu + Fw$,where state variables are subject to additive noises and is analogous to [Rantzer, 2006, Theorem 1] with three key differences:

(1) the control law is the optimal *linear* law i.e., we assume linearity whereas linearity is proved in Rantzer [2006];
(2) state variables are subject to additive white Gaussian channel-noise e in the feedback path (as distinct from the usual process noise w) and as the solution of the optimization problem establishes the respective cross and auto-covariances of w, x, e, u, the per-state SNRs

$$\lambda_i = \frac{\mathbf{E}\left\{x_i^2\right\}}{\mathbf{E}\left\{e_i^2\right\}} \tag{5}$$

are easily found; and
(3) the cost function now includes a term that is SNR-dependent i.e., dependent on $\lambda = \text{diag}\{\lambda_1, \ldots, \lambda_{n_x}\}$.

Theorem 2.1 *Suppose $A \in \mathbb{R}^{n_x \times n_x}, B \in \mathbb{R}^{n_x \times m_x}, C \in \mathbb{R}^{p_x \times n_x}, D \in \mathbb{R}^{p_x \times m_x}$ and $R_j \in \mathbb{R}^{n_x \times m_x}$ for $1 \leq j \leq J$. Then for every $\delta \geq 0$, the following two statements are equivalent:*

(i) There exists $\gamma \geq 0$, a matrix $K \in \mathbb{R}^{n_x \times m_x}$ and $Q \in \mathbb{R}^{n_x \times n_x}$ such that the

stochastic difference equation

$$x(k+1) = Ax(k) + Bu(k) + Fw(k) \tag{6}$$
$$u(k) = K(x(k) + e(k)) \tag{7}$$

has a stationary zero-mean solution satisfying

$$\gamma \geq \mathbf{E}|Cx + D\tilde{u}|^2 \tag{8}$$
$$Q \geq \mathbf{E}\{xx^T\}\mathbf{E}\{ee^T\}^{-1}\mathbf{E}\{xx^T\} \tag{9}$$
$$Q \geq \mathbf{E}\{x\tilde{u}^T\}\mathbf{E}\{\tilde{e}\tilde{e}^T\}^{-1}\mathbf{E}\{\tilde{u}x^T\} \tag{10}$$
$$\delta \geq \alpha|Q| + \gamma \tag{11}$$
$$\mathbf{E}\{x^T R_j \tilde{u}\} = 0 \quad \forall\, 1 \leq j \leq J\,, \tag{12}$$

where $\tilde{u}(k) = Kx(k)$, $\tilde{e}(k) = Ke(k)$, $w \in \mathbb{R}^n$ *is Gaussian white noise with unit variance with* $w(k)$ *is independent of* $x(j)$ *for* $j \leq k$ *and* $e \in \mathbb{R}^{n_x}$ *is a zero-mean Gaussian noise signal independent of* x *and* w *such that* $\mathbf{E}\{ee^T\} > 0$.

(ii) There exists an $X \in \mathbb{R}^{(2n_x+m_x)\times(2n_x+m_x)}$ *with* $X \geq 0$ *given by*

$$X = \begin{bmatrix} X_{xx} & X_{x\tilde{u}} & 0 \\ X_{\tilde{u}x} & X_{\tilde{u}\tilde{u}} & 0 \\ 0 & 0 & X_{\tilde{e}\tilde{e}} \end{bmatrix}$$

and $0 < X_{ee} \in \mathbb{R}^{n_x \times n_x}$ *such that*

$$X_{xx} = [A \quad B]\tilde{X}[A \quad B]^T + BX_{\tilde{e}\tilde{e}}B^T + FF^T \tag{13}$$
$$\gamma \geq \operatorname{tr}\left([C \quad D]\tilde{X}[C \quad D]^T\right) \tag{14}$$
$$0 \leq \begin{bmatrix} X_{ee} & X_{xx} \\ X_{xx} & Q \end{bmatrix} \tag{15}$$
$$0 \leq \begin{bmatrix} Q & X_{x\tilde{u}} \\ X_{\tilde{u}x} & X_{\tilde{e}\tilde{e}} \end{bmatrix} \tag{16}$$
$$0 \leq (\delta - \gamma)I - \alpha Q \tag{17}$$
$$0 = \operatorname{tr}(X_{\tilde{u}x}R_j) \quad \forall\, 1 \leq j \leq J\,, \tag{18}$$

where

$$\tilde{X} = \begin{bmatrix} X_{xx} & X_{x\tilde{u}} \\ X_{\tilde{u}x} & X_{\tilde{u}\tilde{u}} \end{bmatrix}.$$

Moreover, if X *satisfies the conditions of (ii) for the minimal feasible* δ, *then* (6)-(7) *with the linear control law*

$$\mu(\xi) = X_{\tilde{u}x}X_{xx}^{-1}\xi$$

has a solution satisfying (8)-(12). *If* $\tilde{X} > 0$, *then the control law is also stabilizing.*

Proof. The implication $(i) \Rightarrow (ii)$ follows from the conditions of (i) when we define $X_{ee} = \mathbf{E}\{ee^T\}$,

$$X = \begin{bmatrix} X_{xx} & X_{x\tilde{u}} & 0 \\ X_{\tilde{u}x} & X_{\tilde{u}\tilde{u}} & 0 \\ 0 & 0 & X_{\tilde{e}\tilde{e}} \end{bmatrix} = \mathbf{E} \begin{bmatrix} x \\ \tilde{u} \\ \tilde{e} \end{bmatrix} \begin{bmatrix} x \\ \tilde{u} \\ \tilde{e} \end{bmatrix}^T ,$$

and since w has identity covariance. Let $\mu(\xi) = L\xi$, where L is given by $L = X_{\tilde{u}x}X_{xx}^{-1}$. For the implication $(ii) \Rightarrow (i)$, we assume that the inequalities of (ii) hold with the prescribed control. Since $\tilde{X} \geq 0$ we have that

$$X_{\tilde{u}\tilde{u}} \geq X_{\tilde{u}x}X_{xx}^{-1}X_{x\tilde{u}} \,. \tag{19}$$

and equality can be assumed without restriction since reducing $X_{\tilde{u}\tilde{u}}$ can only reduce δ in (17).

Similarly, (15) implies $Q \geq X_{xx}X_{ee}^{-1}X_{xx}$ and equality can be assumed without restriction as reducing Q can only reduce δ in (17) and, hence $Q^{-1} = X_{xx}^{-1}X_{ee}X_{xx}^{-1}$. The inequality imposed by (16) implies that $Q \geq X_{x\tilde{u}}X_{\tilde{e}\tilde{e}}^{-1}X_{\tilde{u}x}$ and, again, we can assume equality without restriction. Combining with (2) we have

$$X_{\tilde{u}x}X_{xx}^{-1}X_{ee}X_{xx}^{-1}X_{x\tilde{u}} = X_{\tilde{e}\tilde{e}} \,. \tag{20}$$

With the control law gain L, (6) is given by $x(k+1) = (A+BL)x(k) + BLe(k) + Fw(k)$. Let $x(0)$ be a Gaussian random variable with $\mathbf{E}\{x(0)\} = 0$ and $\mathbf{E}\{x(0)x(0)^T\} = X_{xx}$. Then, by linearity, $x(1)$ is also Gaussian with $\mathbf{E}\{x(1)\} = 0$ and

$$\begin{aligned}
&\mathbf{E}\{x(1)x(1)^T\} \\
&= (A+BL)\mathbf{E}\{x(0)x(0)^T\}(A+BL)^T + \\
&BL\mathbf{E}\{e(0)e(0)^T\}L^TB^T + FF^T + (A+BL)\mathbf{E}\{x(0)e(0)^T\}L^TB^T \\
&\qquad + BL\mathbf{E}\{e(0)x(0)^T\}(A+BL)^T \\
&= (A+BL)X_{xx}(A+BL)^T + BLX_{ee}L^TB^T + FF^T \\
&\text{by independence of } x \text{ and } e \\
&= AX_{xx}A^T + AX_{x\tilde{u}}B^T + BX_{\tilde{u}x}A^T + BX_{\tilde{u}x}X_{xx}^{-1}X_{x\tilde{u}}B^T \\
&\qquad + BX_{\tilde{u}x}X_{xx}^{-1}X_{ee}X_{xx}^{-1}X_{x\tilde{u}}B^T + FF^T \\
&= AX_{x\tilde{u}}A^T + AX_{x\tilde{u}}B^T + BX_{\tilde{u}x}A^T + BX_{\tilde{u}\tilde{u}}B^T \\
&\quad + BX_{\tilde{e}\tilde{e}}B^T + FF^T \quad \text{by (20)} \\
&= [A \quad B]\tilde{X}[A \quad B]^T + BX_{\tilde{e}\tilde{e}}B^T + FF^T = X_{xx} \,,
\end{aligned}$$

hence, the process is stationary with $\mathbf{E}\{xx^T\} = X_{xx}$. Equations (8)-(12) follow from (14)-(18), respectively.

Additionally, the resultant control law is stabilizing (in the ordinary sense for $w = 0, e = 0$) if $X > 0$ with the quadratic Lyapunov function $V = x^T X_{xx} x$. ■

3 Distributed Control With Covariance Constraints

3.1 Control Synthesis

In view of the discussion in Section 1 and suitable augmentation of the state-space, Theorem 2.1 can be used to solve the distributed control synthesis problem for SNR-constrained communication channels.

Consider a directed graph with a set of M nodes (vertices) $\mathcal{V}$ and a set of edges $\mathcal{E}$,

$$(\mathcal{V}, \mathcal{E}), \quad \mathcal{V} = \{1, \dots, M\}. \tag{21}$$

Suppose J nodes contain controllers u_i. Let (i, j) denote the directed edge from node i to node j. We consider matrices $\Phi = (\Phi_{ij}) \in \mathbb{R}^{n \times n}$ where the $n_i \times n_j$ block Φ_{ij} is zero unless $(i, j) \in \mathcal{E}$. This enforces the property that the network propagates state and controller information at least as fast as the plant and thus no signaling incentive exists. For instance:

$$\Phi = \begin{bmatrix} \Phi_{11} & 0 & \Phi_{13} & 0 \\ \Phi_{21} & \Phi_{22} & \Phi_{23} & 0 \\ 0 & \Phi_{32} & \Phi_{33} & \Phi_{34} \\ 0 & 0 & \Phi_{43} & \Phi_{44} \end{bmatrix}$$

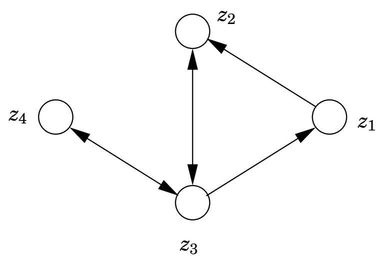


Fig. 2. Compatible Φ and information structure.

Let $\Gamma = [\Gamma_{*1} \cdots \Gamma_{*J}] = (\Gamma_{ij}) \in \mathbb{R}^{n \times m}$ where the $n_i \times m_j$ block Γ_{ij} is zero unless $i = j$. Assume that (Φ, Γ) is stabilizable and

$$z(t+1) = \Phi z(t) + \Gamma u(t) + w(t) \tag{22}$$

where $w(t) = [w_1(t) \cdots w_n(t)]^T$ are zero-mean and Gaussian with $\mathbf{E}\{ww^T\} = I$. Let

$$z = \begin{bmatrix} z_1 \\ \vdots \\ z_J \end{bmatrix} \; u = \begin{bmatrix} u_1 \\ \vdots \\ u_J \end{bmatrix} \; z_i^\dagger(t) = \begin{bmatrix} z_i(t) \\ z_i(t-1) \\ z_i(t-2) \\ \vdots \end{bmatrix} \; e_i^\dagger(t) = \begin{bmatrix} e_i(t) \\ e_i(t-1) \\ e_i(t-2) \\ \vdots \end{bmatrix}$$

Let $v_1, v_2 \in \mathcal{V}$ and let $d(v_1, v_2)$ denote the shortest path from v_1 to v_2. In particular,

$$d(v_1, v_1) = 0 \quad d(v_1, v_2) = 1 \Rightarrow (v_1, v_2) \in \mathcal{E}. \tag{23}$$

When v_1, v_2 are not connected, we adopt the convention that $d(v_1, v_2) = n$, where n is the dimension of the z state-space. Enumerating all vertices in graph with a labeling $v \mapsto k \in \{1, \ldots, M\}$, we adopt the shorthand $d_{ij} = d(i, j)$. Vertices are usually, but not always, associated with particular state variables or control. However, every state variable z_i and each control u_j is associated with some vertex k. Hence, for every pair (z_i, k) (or (u_j, k')), let $\pi : \{z_1, \ldots, z_n, u_1, \ldots, u_J\} \to \{1, \ldots, M\}$ denote the map $z_i \mapsto k$ (or $u_j \mapsto k'$). That is, for each state variable z_i or control u_j, π recovers the associated vertex; i.e., $\pi(z_i) = k, \pi(u_j) = k'$.

The following theorem, stated without proof, establishes that the optimal distributed control synthesis problem can be solved with finite capacity utilization for an appropriate class of information structures. Essentially, it is an application of Theorem 2.1 for appropriately chosen matrices.

Theorem 3.1 *The following statements are equivalent for* (22)*:*

(i) There exist (linear) feedback laws

$$\begin{aligned} u_j(t) = \mu_j(z_1^\dagger(t - d_{\tilde{j}\pi(z_1)}) + e_1^\dagger(t - d_{\tilde{j}\pi(z_1)}), \\ \ldots, z_n^\dagger(t - d_{\tilde{j}\pi(z_n)}) + e_n^\dagger(t - d_{\tilde{j}\pi(z_n)}))\,, \end{aligned} \tag{24}$$

where $\tilde{j} = \pi(u_j)$*, that together with* (22) *have a stationary zero-mean solution satisfying*

$$\begin{aligned} \delta &\geq |\mathbf{E}\,\{x\tilde{u}^T\}\mathbf{E}\{\tilde{e}\tilde{e}^T\}^{-1}\mathbf{E}\{\tilde{u}x^T\}| + \mathbf{E}|\tilde{C}z + D\tilde{u}|^2 \\ \delta &\geq |\mathbf{E}\,\{xx^T\}\mathbf{E}\,\{ee^T\}^{-1}\mathbf{E}\,\{xx^T\}| + \mathbf{E}|\tilde{C}z + D\tilde{u}|^2 \end{aligned}$$

(ii) There exists a (linear) feedback law $u(k) = \mu(x + e)$ *that together with* (22) *has a stationary zero-mean solution satisfying*

$$\begin{aligned} \delta &\geq |\mathbf{E}\,\{x\tilde{u}^T\}\mathbf{E}\{\tilde{e}\tilde{e}^T\}^{-1}\mathbf{E}\{\tilde{u}x^T\}| + \mathbf{E}|\tilde{C}z + D\tilde{u}|^2 \\ \delta &\geq |\mathbf{E}\,\{xx^T\}\mathbf{E}\,\{ee^T\}^{-1}\mathbf{E}\,\{xx^T\}| + \mathbf{E}|\tilde{C}z + D\tilde{u}|^2 \\ 0 &= \mathbf{E}\,\{u_i(k)w_j(k - l)\} \end{aligned}$$

for $1 \leq j \leq J$ *and* $1 \leq l \leq d_{ij}$*, where* $\tilde{e}, \tilde{u}, x$ *are defined as in Section 2.*

Remark 1 *Where every vertex corresponds to a state-variable, the most natural labeling of the graph is* $\{z_1, \ldots, z_n\} \mapsto \{1, \ldots, n\}$*. In other cases, and with reference to* (24)*, the map* π *is needed to recover vertices from particular state variables or controls.*

For appropriately chosen matrices R_j, the covariance constraints $\mathbf{E}\{u_i(k)w_j(k-l)\} = 0$ can be written in the form $\mathrm{tr}(X_{\tilde{u}x}R_j)$, as discussed in Section 1. Hence, Theorem 2.1 establishes the existence of a linear feedback law with a stationary solution satisfying the covariance constraints. As shown in the proof of Theorem 2.1, the closed loop is exponentially stable and thus the control law can be written in the form

$$u_i(k) = \sum_{m=0}^{\infty} K_{ijm}(z_j(k-m) + e_j(k-m)). \tag{25}$$

The covariance constraints show that $K_{ijm} = 0$ for $m < d_{ij}$ and the conditions of (ii) are met. In fact, in view of the relationship of the augmented state variable x to z, we only need consider finite sums in (25) where the upper terminal is given by $\max_{i,j\in\mathcal{V}} d_{ij} + 1$.

3.2 Allocating Link Capacity

If the LMI in Theorem 3.1 is feasible, then as in Braslavsky et al. [2004], we assume that the respective channels are not band-limited and, asymptotically, the capacities are given by

$$C_i = \frac{\lambda_i}{2\ln 2}, \tag{26}$$

where λ_i denote the ith SNR.

For a given directed *network graph*, these data-rates or *flows* can be allocated optimally along edges or links of the network by solving an appropriate multi-commodity graph flow problem:

(1) Each control u_j is a consumer of information flows of state variables that it depends on (commodities). In particular, any u_j consumes z_i at a rate of C_i.
(2) Each vertex of the network graph associated with a state variable z_i is a producer of a commodity with an output rate of $J_i \cdot C_i$, where J_i is the number of controllers that consume z_i.

Condition 2 asserts that each controller "gets its own copy" of state variable flows that it consumes. Enumerating the vertices of the network graph by $\mathcal{V} = \{1, \ldots, M\}$, denoting the edges of the graph by $\mathscr{E}$, the set of commodities $\mathcal{R} = \{1, \ldots, n\}$ and, denoting the directed flows of commodity r from i to j by f_{ij}^r, we have

$$\sum_{j\in\mathrm{Out}(i)} f_{ij}^r + \mathscr{D}_i^r = \sum_{j\in\mathrm{In}(i)} f_{ji}^r + \mathscr{S}_i^r \quad r \in \mathcal{R}, i \in \mathcal{V}, \tag{27}$$

where $\mathrm{Out}(i) = \{j : (i,j) \in \mathscr{E}\}$, $\mathrm{In}(i) = \{j : (j,i) \in \mathscr{E}\}$ and

$$\mathscr{D}_i^r = \begin{cases} C_i & \text{if } i \text{ is a consumer of } r \\ 0 & \text{otherwise} \end{cases} \tag{28}$$

$$\mathscr{S}_i^r = \begin{cases} J_i C_i & \text{if } i \text{ is a producer of } r \\ 0 & \text{otherwise.} \end{cases} \tag{29}$$

Note that (27) is a linear (equality) constraint on any graph flow optimization problem. Let $f_{ij} = \sum_{r\in\mathcal{R}} f_{ij}^r$ and we adopt the convention that $f_{ii} = 0$. The capacity allocation problem becomes:

$$\text{minimize} \sum_{i,j\in\mathcal{V}} c_{ij} f_{ij} \text{ subject to} \tag{30}$$

$$f_{ij} \geq 0, \quad i,j \in \mathcal{V} \tag{31}$$

and the conservation of flow constraint (27).

4 Case Studies & Comparisons

This section will consider the following five-node system:

$$z_i^+ = i z_i + u_i + w_i \quad i \in \{1,\ldots,5\}, \tag{32}$$

where $\mathbf{E}\{ww^T\} = I$ together with the quadratic-cost function

$$\delta = \mathbf{E}(z^T P z + u^T N u) \tag{33}$$

as the subject of discourse to explore several features of the results of this paper, where

$$N = P = [p_{ij}] = \begin{bmatrix} 0.6222 & -1.1103 & -0.3899 & -0.4816 & -0.1399 \\ -1.1103 & 24.8482 & 9.0286 & 13.1770 & -2.7969 \\ -0.3899 & 9.0286 & 4.9151 & 6.7254 & -0.0525 \\ -0.4816 & 13.1770 & 6.7254 & 9.4213 & -0.3244 \\ -0.1399 & -2.7969 & -0.0525 & -0.3244 & 1.1443 \end{bmatrix}. \tag{34}$$

Ignoring issues of channel-capacity momentarily, there are two extremes with respect to the information structure of the system:

(a) nodes are disconnected, that is, for each node i, $u_i(k)$ is a function of $z_i(k)$ only;

(b) nodes are completely connected with a delay-free network graph, that is, for each node i, $u_i(k)$ is a function of $z_1(k), \ldots, z_5(k)$.

Note that both of these information structures are admissible as the system is decoupled. The optimal control law is in fact linear and can be found via standard LQG theory in both cases. For systems of the form:

$$z^+ = \Phi z + \Gamma u + w \,, \tag{35}$$

the optimal control for costs of the form (33) is given by

$$L = (N + \Gamma^T S \Gamma)^{-1} (\Gamma^T S \Gamma) \,,$$

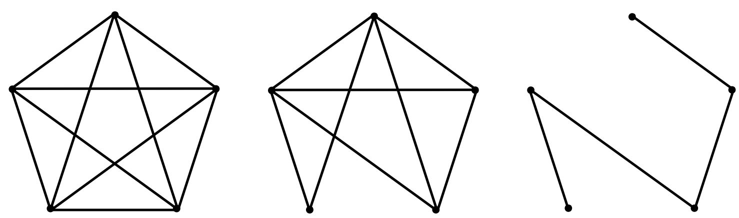
where S solves the algebraic Ricatti equation (ARE)

$$S = \Phi^T S \Phi + PL(N + \Gamma S^T \Gamma) L \,. \tag{36}$$

The minimal cost is given by $\delta = \mathrm{tr}(S)$. See e.g., [Söderström, 2002, Chapter 11], for further details.

(a) When nodes are disconnected, the covariances in (33) vanish and we can consider the individual costs $\delta_i = \mathbf{E}(z_i^T p_{ii} z_i + u_i^T p_{ii} u_i)$, hence, we solve a scalar ARE (36) for each i, with $\Phi_i = i$. The minimum cost is given by $\delta = \sum_i \delta_i = 331.02$.
(b) When nodes are completely connected, (36) is a 5×5 equation, $\Phi = \mathrm{diag}\{1, \ldots, 5\}$ and the cost (33) with P and N as in (34). The minimal cost is given by $\delta = 249.44$.

We have that complete connectivity reduces the cost which is in line with intuition. The capacity-constrained case is now examined for the network graph structures shown in Figure 3(a), Figure 3(b) and Figure 3(c) for various information structures and parameters of the cost function. Specifically, we will be concerned with the "classical" information structure – availability of all state variables at each controller without delay – as well as various restrictions of the availability of state variables at each controller.



(a) (b) (c)

Fig. 3. Full (strong connectedness) (a), partial (b) and minimal (c) interconnectivity.

A summary of results are presented in Figure 4. Notably, the costs δ, γ and capacities C_i are independent of the network structure. This was emphasized earlier *per-state capacities covariance costs do not depend on the network structure.* They are the result of the application of Theorem 3.1, not of solving the graph flow problem. Figure 4 reveals that the lowest δ-cost solutions do not necessarily yield the lowest total flow, and hence, capacity required in the network and the converse is also true.

$\alpha = 0.1$	
Network Graph Figure 3(a)	$C_i = \{3.36, 2.88, 3.83, 2.22, 1.85\} \times 10^{-4}$ $\sum_{i,j\in\mathcal{V}} f_{ij} = 0.00565, \delta = 321.95, \gamma = 293.96$
Network Graph Figure 3(b) u_1 independent of z_2 u_2 independent of z_1	$C_i = \{4.60, 6.94, 9.05, 5.49, 4.37\} \times 10^{-4}$ $\sum_{i,j\in\mathcal{V}} f_{ij} = 0.0110, \delta = 323.83, \gamma = 296.46$
Network Graph Figure 3(c) u_i depends only $z_{\max\{i-1,1\}}$, z_i and $\mathrm{z}_{\min\{i+1,5\}}$	$C_i = \{1.16, 1.55, 5.23, 1.75, 1.13\} \times 10^{-4}$ $\sum_{i,j\in\mathcal{V}} f_{ij} = 0.0019, \delta = 351.11, \gamma = 327.04$

Fig. 4. Summary of costs and capacities for the design case study.

Figure 4 does not illustrate the individual flows along edges of the graph. Let $f_{ij} = \sum_{r\in\mathcal{R}} f_{ij}^r$. The directional flows resultant from network graph Figure 3(c) are given by

$$[f_{ij}] = 10^{-3} \times \begin{bmatrix} 0 & 0.1162 & 0 & 0 & 0 \\ 0.1548 & 0 & 0.1548 & 0 & 0 \\ 0 & 0.5234 & 0 & 0.5234 & 0 \\ 0 & 0 & 0.1752 & 0 & 0.1752 \\ 0 & 0 & 0 & 0.1130 & 0 \end{bmatrix} \text{bps.}$$

These flows are not identical in different directions and obviously vary by large relative amounts along different edges of the graph. This is an indication that using a single figure of merit e.g., a single capacity figure, within an NCS is an inaccurate reflection of capacity requirements and that different edges (links) have non-uniform capacity requirements.

References

Bassam Bamieh and Peter G. Voulgaris. A convex characterization of distributed control problems in spatially invariant systems with communication constraints. *Sys. Contr. Lett.*, 54(6):575–583, June 2005.

Bassam Bamieh, Fernando Paganini, and Munther A. Dahleh. Distributed control of spatially invariant systems. *IEEE Trans. Automat. Contr.*, 47(7): 1091–1107, July 2005.

S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*, volume 15 of *Studies in Applied Mathematics*. SIAM, 1995.

J.H Braslavsky, R.H. Middleton, and J.S. Freudenberg. Feedback stabilization over signal-to-noise ratio constrained channels. In *American Control Conference*, volume 6, pages 4903–4908, 2004.

Chandra Chekuri, Sanjeev Khanna, and F. Bruce Shepherd. The all-or-nothing multicommodity flow problem. In *STOC '04: Proceedings of the thirty-sixth annual ACM symposium on Theory of computing*, pages 156–165, New York, NY, USA, 2004. ACM Press. ISBN 1-58113-852-0. doi: http://doi.acm.org/10.1145/1007352.1007383.

Y-C. Ho and K.C. Chu. Team decision theory and information structures in optimal control problems – Part 1. *IEEE Trans. Automat. Contr.*, 39(10): 15–22, 1972.

Daniel Liberzon and J. P. Hespanha. Stabilization of nonlinear systems with limited information feedback. *IEEE Trans. Automat. Contr.*, 50(6):910–915, 2005.

Nuno C. Martins. Witsenhausen's counter example holds in the presence of side information. In *Proc. Conf. Decis. Contr.*, pages 1111–1116, San Diego, CA, December 2006.

G. N. Nair, R. J. Evans, I. M. Y. Mareels, and W. Moran. Topological feedback entropy and nonlinear stabilization. *IEEE Trans. Automat. Contr.*, 49(9): 1585–1597, 2004.

G.N. Nair and R.J. Evans. Stabilizability of stochastic linear systems with finite feedback data rates. *SIAM J. Contr. Optimization*, 43(2):413–436, July 2004.

G.N. Nair, F. Fagnani, S. Zampieri, and R.J. Evans. Feedback Control under Data Rate Constraints: an Overview. *to appear in IEEE Proc.*, 2007.

Anders Rantzer. Linear quadratic team theory revisited. In *American Control Conference*, Minneapolis, MN, Jun 2006.

M. Rotkowitz and S. Lall. Decentralized control information structures preserved under feedback. In *Proc. Conf. Decis. Contr.*, pages 569–575, Las Vegas, NV, December 2002.

Torsten Söderström. *Discrete-time Stochastic Systems*. Springer, 2nd edition, 2002.

H.S. Witsenhausen. A counterexample in stochastic optimum control. *SIAM J. Contr.*, 6(1):131–147, 1968.